Math. Proc. Camb. Phil. Soc. (1992), **112**, 127
Printed in Great Britain

Discriminants of Casson–Gordon invariants

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(*Received* 13 *May* 1991)

0. *Introduction*

Casson–Gordon invariants were first used to prove that certain algebraically slice knots in S^3 are not slice knots [**2**, **3**]. Since then they have been applied to a wide range of problems, including embedding problems and questions relating to boundary links [**2**, **10**, **21**, **25**]. The most general Casson–Gordon invariant takes its value in $L_0(\mathbb{Q}(\zeta_d)(t)) \otimes \mathbb{Q}$; here ζ_d denotes a primitive dth root of unity. Litherland [**20**] observed that one could usually tensor with $\mathbb{Z}_{(2)}$ instead of $\mathbb{Q}$, and in this way preserve the 2-torsion in the Witt group. He then constructed new examples of non-slice genus two knots which were detected with torsion classes in $L_0(\mathbb{Q}(\zeta_d)(t)) \otimes \mathbb{Z}_{(2)}$ modulo the image of $L_0(\mathbb{Q}(\zeta_d)) \otimes \mathbb{Z}_{(2)}$. His examples necessarily have genus greater than one. We present genus one examples which are detected by torsion in $L_0(\mathbb{Q}(\zeta_d)) \otimes \mathbb{Z}_{(2)}$. Most importantly, we relate that torsion to torsion in the algebraic knot concordance group, G_-, defined by Levine [**18**]. These techniques are then applied in a variety of other situations, including the question of when a two component link is concordant to a boundary link, the study of $\mathbb{Z}_p$-homology cobordism of 3-manifolds, and the problem of which 3-manifolds embed in 4-space.

A knot in S^3 is a slice knot if it extends to a locally flat embedding of a 2-disc in the 4-ball. Suppose that K has a genus one Seifert surface F. By a surgery curve for F we mean a simple closed curve J which is not null-homologous on F and for which a parallel curve to J on F has linking number zero with J. If a surgery curve for F is itself a slice knot, then one may perform surgery on F in the 4-ball to obtain a slice disc for K. A theorem of Levine states that if K is a slice knot, then it has a surgery curve (although there is no guarantee that one can obtain a slice disc by surgery along a surgery curve). If F has a surgery curve then K is said to be algebraically slice. It is not hard to see that if F has a surgery curve, then it has exactly two surgery curves up to isotopy and change of orientation.

Gilmer's work in [**7**] demonstrated that the Casson–Gordon invariants (for the most relevant characters) of an algebraically slice knot could be interpreted in terms of Tristram–Levine signatures of surgery curves. In this way Casson–Gordon invariants could be interpreted as secondary obstructions to a knot being slice. The new torsion Casson–Gordon invariants have the same interpretation. In [**10**] it was shown that the general Witt class Casson–Gordon invariant can, at times, be interpreted in terms of Witt class invariants of knots on the Seifert surface. This was extended in [**8**]. All the examples in these papers could be detected by signatures alone. Here we will present examples that cannot be detected with signatures, since the relevant knots on the Seifert surfaces represent torsion in the algebraic knot concordance group.

Let $B_m(K)$ denote the m-fold branched cyclic cover of S^3 along K. If m is a prime power $B_m(K)$ is a $\mathbb{Z}_m$-homology sphere. By a theorem of Plans [**11**, **24**], if m is odd, $H_1(B_m(K))$ is a direct double. Let d denote an odd prime power, *a convention that holds throughout this paper*. Define $\Delta_d(K)$ to be the square root of the order of $H_1(B_d(K))$. We will say an integer n is a d-norm if it is positive and if every prime factor of n which is relatively prime to d and has odd exponent in n, has odd order in $\mathbb{Z}_d^*$. By Proposition 4 below, if K is algebraically slice, then $\Delta_d(K)$ is a d-norm. If K is algebraically slice then the order of $H_1(B_2(K))$ is an odd square, say $(2m(K)+1)^2$ where $m(K)$ is greater than or equal to zero. The following is a corollary of our work which is easy to state (and apply).

COROLLARY 1. *Let K be a slice knot with a genus one Seifert surface F. Then one of the two surgery curves for F, say J, satisfies the following property: if q is an odd prime power, and d divides $(m(K)+1)^q-(m(K))^q$, then $\Delta_d(J)$ is a d-norm.*

Thus if the fact that surgery curves for K are non-slice can be detected by Δ_d failing to be a d-norm (for one of certain specified d's depending on $m(K)$), then the corollary says K must fail to be slice as well. Let $J(m/q)$ denote m/q surgery to S^3 along a knot J. We also have

COROLLARY 2. *Let m be odd, d be a divisor of m and N be a $\mathbb{Z}_d$-homology 3-sphere. If $J_1(m/q_1)\# J_2(m/q_2)\# N$ embeds in S^4, then $\Delta_d(J_1)\Delta_d(J_2)$ is a d-norm.*

The paper is organized as follows. Section 1 provides background material and develops the algebra needed for the later sections. In Section 2 we present the basic applications of discriminants of Casson–Gordon invariants to knot and link concordances. Section 3 studies the application of these methods to problems concerning the embedding of 3-manifolds in 4-space, and more generally to the study of homology cobordisms of 3-manifolds. Section 4 is devoted to developing methods for efficiently computing $\Delta_d(J)$. We concentrate on the case when J is a k-twisted double of the unknot. The doubles of the unknot are of particular interest in that they provided the first class of knots representing 2-torsion and 4-torsion in the algebraic knot concordance group [**19**], and were also among Casson and Gordon's first examples of non-slice algebraically slice knots [**3**]. We use them as building blocks for most of our examples. Section 5 computes the discriminant of lens spaces.

1. *Preliminaries*

In [**10**] we defined a simple Witt class invariant $w_{i/m}(J)$ for a knotted $(2n-1)$-sphere J, where $0<i<m$, and m is odd. If n is odd, $w_{i/m}(J)\in L_0(\mathbb{Q}(\zeta_m))\otimes\mathbb{Z}_{(2)}$ and if n is even $w_{i/m}(J)\in L_2(\mathbb{Q}(\zeta_m))\otimes\mathbb{Z}_{(2)}$. If m is a prime power, this is a well defined invariant of the algebraic concordance class of the knot. Let N_m denote the quotient group of $\mathbb{Q}(\zeta_m+\bar{\zeta}_m)$ modulo the subgroup of norms of elements in $\mathbb{Q}(\zeta_m)$. One defines the discriminant $\operatorname{dis}: L_0(\mathbb{Q}(\zeta_m))\to N_m$ by $\operatorname{dis}[A]=(-1)^{r(r-1)/2}\det(A)$, where A is an $r\times r$ Hermitian matrix over $\mathbb{Q}(\zeta_m)$. The discriminant is a homomorphism on the ideal with representatives of even rank [**1**, **5**]. The map from $L_2(\mathbb{Q}(\zeta_m))$ to $L_0(\mathbb{Q}(\zeta_m))$ which

sends a form given by a skew-Hermitian matrix A to the form given by the Hermitian matrix $(\zeta_m-\bar{\zeta}_m)A$ is a homomorphism. We define dis on $L_2(\mathbb{Q}(\zeta_m))$ via this homomorphism and we define $\delta_{i/m}(K)\in N_m$ to be $\operatorname{dis}(w_{i/m}(K))$.

Let $\Delta_J(t)$ denote the symmetrized normalized representative of the Alexander polynomial defined as follows. Let V be a $2g\times 2g$ Seifert matrix for J. If n is odd define $\Delta_J(t)$ by $(t)^{-g}(\det(tV-V^t))$. If n is even, define $\Delta_J(t)$ to be

$$(-1)^g t^{-g}(\det(tV+V^t)).$$

One may check that these are invariants of the S-equivalence class of J as in [**16**]. In either case, it follows from the formulae in appendix 2 of [**10**] that

$$\delta_{i/d}(J)=\Delta_J(\zeta_d^i)$$

in N_d. (We chose our normalization so that this last equation would hold.) Note that $\delta_{i/d}(J)=\delta_{-i/d}(J)$.

According to a formula due to Fox for classical knots, $|H_1(B_m(J))|$ is given by $|\prod_{i=1}^{m-1}\Delta_J(\zeta_m^i)|$ (see [**12**], p. 17). Note that for m odd the second absolute value sign is unnecessary and that $\Delta_m(J)$ is the square root of this order. We extend our definition of $\Delta_m(J)$ to arbitrary dimension and general odd m by letting $\Delta_m(J)$ denote the square root of $\prod_{i=1}^{m-1}\Delta_J(\zeta_m^i)$, or, equivalently, the absolute value of $\prod_{i=1}^{\frac{1}{2}(m-1)}\Delta_J(\zeta_m^i)$. This is an integer as it is the product (over all divisors m' of m) of the norms of $\Delta_J(\zeta_{m'})$ in $\mathbb{Z}[\zeta_{m'}+\overline{\zeta_{m'}}]$, the ring of integers of $\mathbb{Q}[\zeta_{m'}+\overline{\zeta_{m'}}]$, under the norm of the extension $\mathbb{Q}[\zeta_{m'}+\overline{\zeta_{m'}}]$ over $\mathbb{Q}$. In Section 4 we will relate $\Delta_m(J)$ in the non-classical case to the homology of $B_m(J)$, the m-fold branched cyclic cover of S^{2n+1} along J, assuming that J is simple.

We have the following lemma from algebraic number theory, which explains our terminology 'd-norm'. We will only use the 'if' part, which is the more elementary fact.

LEMMA 3. *An integer is a d-norm if and only if it can be written in the form $z\bar{z}$ where $z\in\mathbb{Q}(\zeta_d)$.*

Proof. Let n be an integer and suppose that $n=z\bar{z}$ where $z\in\mathbb{Q}(\zeta_d)$. There exists a k in $\mathbb{Z}$ such that $kz=\nu\in\mathbb{Z}(\zeta_d)$, so that $nk^2=\nu\bar{\nu}$. Let p be a prime relatively prime to d and of even order in $\mathbb{Z}_d^*$. We must show it has even exponent in the prime factorization of n (or equivalently nk^2) in $\mathbb{Z}$. Consider first the factorization of the ideal $p\mathbb{Z}(\zeta_d)=Q_1Q_2\ldots Q_r$ as a product of distinct prime ideals in $\mathbb{Z}(\zeta_d)$ (see [**22**], theorem 26). The Galois group, which is cyclic of order $\phi(d)$, acts transitively on the Q_i (see [**22**], theorem 23). The isotropy subgroup of each Q_i is then cyclic of order $\phi(d)/r=f$, where f is the order of p in $\mathbb{Z}_d^*$. As the isotropy subgroup has even order, it must include complex conjugation, the unique element of order 2 in the Galois group. Thus $Q_i=\bar{Q}_i$ for each i. If p has odd exponent in nk^2, then each Q_i has odd exponent in the prime factorization of $(nk^2)\mathbb{Z}(\zeta_d)$. But this exponent is simply twice the exponent of Q_i in $\nu\mathbb{Z}(\zeta_d)$.

To prove the converse it is enough to show that if a prime p is a d-norm, then it can be written in the form $z\bar{z}$ where $z\in\mathbb{Q}(\zeta_d)$. If d is a power of p, we can let z be $\prod_{i=1}^{\frac{1}{2}(p-1)}(1-\zeta_p^i)$ (see for instance [**22**], exercise 17, p. 9). If p is relatively prime to d, then p is a d-norm if and only if it has odd order in $\mathbb{Z}_d^*$. Thus we must show that if p has odd order in $\mathbb{Z}_d^*$, then it is a norm of the quadratic extension $\mathbb{Q}(\zeta_d)$ over $\mathbb{Q}(\zeta_{d'}+\overline{\zeta_{d'}})$. Again we consider the prime factorization of $p\mathbb{Z}(\zeta_d)$ as an ideal and obtain this time $Q_1 Q_2 \ldots Q_{r/2} \bar{Q}_1 \bar{Q}_2 \ldots \bar{Q}_{r/2}$. Now we make use of the discussion in [**1**], pp. 65–66. A prime π in $\mathbb{Q}(\zeta_d+\overline{\zeta_d})$ is called inert if $\pi\mathbb{Z}(\zeta_d)$ is a prime ideal in $\mathbb{Z}(\zeta_d)$, necessarily invariant under complex conjugation. The prime factorization of $p\mathbb{Z}(\zeta_d+\overline{\zeta_d})$ cannot have any inert terms as then the factorization of $p\mathbb{Z}(\zeta_d)$ would have some factors invariant under complex conjugation.

Let σ denote $\zeta_d+\overline{\zeta_d}-2$. By the Hilbert symbol of p at a prime π in $\mathbb{Q}(\zeta_d+\overline{\zeta_d})$ we will mean that Hilbert symbol $(p,\sigma)_\pi$. By the above, it follows that the Hilbert symbols of p at the inert primes vanish. They automatically vanish at the split primes. A positive integer is positive in every ordering, and so the symbols of p at the infinite primes vanish as well. There is only one ramified prime, and so by Hilbert reciprocity, the symbol at this ramified prime vanishes as well. As all the Hilbert symbols of p vanish, p is a norm by the Hasse cyclic norm theorem. ∎

PROPOSITION 4. *If J is algebraically slice, then $\Delta_d(J)$ is a d-norm for all d.*

Proof. It is clear that $w_{i/d}(J)$ is zero, and so $\delta_{i/d}(J) = \Delta_J(\zeta_d^i)$ represents 1 in N_d. Therefore $\Delta_d(J)$ is a norm from $\mathbb{Q}(\zeta_d+\overline{\zeta_d})$, and so by Lemma 3 it is a d-norm. ∎

2. *Concordance to boundary links and slice knots*

In [**10**] we considered a family of links $L_\mu(J)$ in S^{2n+1} with two co-dimension two components, (K_1, K_2), first considered by Cochran and Orr [**4**]. Here, J is a knotted $(2n-1)$-sphere, and μ is an integer. (See the beginning of section 3 of [**10**] for a precise description.) Cochran and Orr state in [**4**] that if J does not represent torsion in the algebraic knot cobordism group and μ is not 0 or -1, then the resulting links are not concordant to boundary links. Below we will obtain the same conclusion if J represents some torsion classes and $\mu = 1$. For $n = 1$, this is the link pictured in Figure 1 with $J_1 = J$ and J_2 unknotted. The $2\mu+1$ half twists in Figure 1 are not meant to introduce twisting on the bands of the evident Seifert surface. The high-dimensional knot K_1 was first studied by Ruberman [**25**]. He was generalizing the knot in figure 5 of [**9**]. In theorem 5 of [**10**] we showed that if this link is concordant to a boundary link, then a certain sum $\sum_{i\in sS} w_{i/d}(J)$ is zero. Taking discriminants we obtain the following:

THEOREM 5. *If $L_\mu(J)$ is concordant to a boundary link, then for each prime power q and prime power d dividing $((\mu+1)^q-\mu^q)$, one has*

$$\prod_{i\in sS} \delta_{i/d}(J) = 1 \in N_d,$$

where S is the multiplicative subgroup generated by $\mu^{-1}(\mu+1)$ in $\mathbb{Z}_d^*$, the group of units in $\mathbb{Z}_d$, and s is an arbitrary element in $\mathbb{Z}_d^*$.

In order to make it easier to determine whether the above product is 1 in N_d, we take a further product (perhaps losing information) so that we can apply Lemma 3. We obtain:

COROLLARY 6. *If $L_\mu(J)$ is concordant to a boundary link, then $\Delta_d(J)$ is a d-norm, for each prime power q and prime power d dividing $((\mu+1)^q-\mu^q)$.*

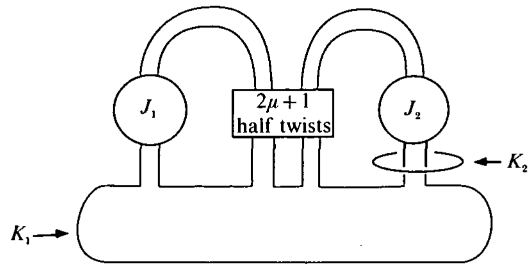

Fig. 1

Proof. First we suppose that d is a prime. As q is odd, and $(\mu^{-1}(\mu+1))^q = 1 \bmod d$, S cannot contain -1. It follows that the cosets sS and $-sS$ are disjoint. Hence it is possible to pick a collection of cosets so that their union contains exactly one element from each of the sets $\{s, -s\}$ in $\mathbb{Z}_d^*$. Now if we take the product of the equalities that the theorem gives us for each of these cosets, and recall that $\delta_{i/d}(K) = \delta_{-i/d}(K)$, we obtain

$$\Delta_d(J) = \prod_{i=1}^{\frac{1}{2}(d-1)} \delta_{i/d}(J) = 1$$

in N_d. If d is a prime power we modify the proof as follows. For each d' which divides d, we take a product $\Pi_{d'}$ over an analogous collection of cosets in $\mathbb{Z}_{d'}^*$. Then

$$\Delta_d(J) = \prod_{d'|d} \Pi_{d'} = 1$$

in N_d as before. ∎

Remark. We may obtain exactly the same conclusion for the link pictured in Figure 1 with $J_1 = J$ and J_2 arbitrary, and K_2 replaced with any component which links the first band algebraically zero times and the second band algebraically non-trivially. This follows from corollary 2·2 of [**8**].

By taking the discriminant of the conclusion of theorem 4 of [**8**] we obtain Theorem 7 below. The proof of Corollary 6 then yields a proof of Corollary 1 mentioned in the Introduction.

THEOREM 7. *Let K be a slice knot with a genus one Seifert surface F. Then one of the two surgery curves for F, say J, has the following property: for each prime power q and prime power d dividing $((m(K)+1)^q - m(K)^q)$, one has*

$$\prod_{i \in sS} \delta_{i/d}(J) = 1 \in N_d,$$

where S is the multiplicative subgroup generated by $(m(K))^{-1}(m(K)+1)$ in $\mathbb{Z}_d^*$, the group of units in $\mathbb{Z}_d$, and s is an arbitrary element in Z_d^*.

Examples. Let T_k be the k-twisted double of the unknot, illustrated in Figure 2. As 2 has order 11 in $\mathbb{Z}_{23}^*$, 23 is a divisor of $2^{11}-1$. By Theorem 14 in Section 4, $\Delta_{23}(T_3) = (137)(277)(5657)$. Since $137 = -1 \bmod 23$ and has odd exponent in $\Delta_{23}(T_3)$, it follows that $\Delta_{23}(T_3)$ is not a 23-norm. Thus the link in Figure 1 with $J_1 = T_3$ is not concordant to a boundary link. Letting J be a knotted $(4r+3)$-sphere with the same Alexander polynomial as T_5, one has that $L_1(J)$ is not concordant to a boundary link.

Similarly the knot K_1 in Figure 1, with $\mu = 1$ and $J_1 = J_2 = T_3$, is not slice. Perhaps a nicer example is the knot K in Figure 3. Again the three half twists are not

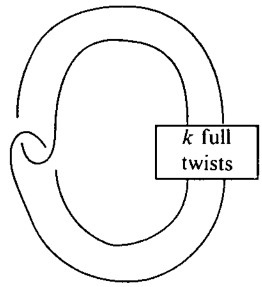

Fig. 2

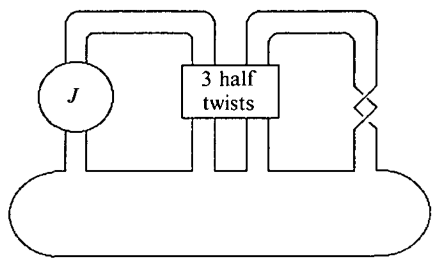

Fig. 3

intended to introduce twists on the bands of the evident Seifert surface. We have $m(K) = 1$, and both surgery curves are isotopic to J. Thus if $J = T_3$, then K is not slice. In Section 4, we will generalize all these examples. We will also give examples of knotted $(4r+1)$-spheres U_k representing torsion classes with $L_1(U_k)$ not concordant to a boundary link.

3. $\mathbb{Z}_p$*-homology cobordism and embedding* 3*-manifolds in* 4*-space*

Throughout this section m will be odd. Let (M,χ) be a 3-manifold M and a homomorphism $\chi: H_1(M) \to \mathbb{Z}_m$. Together M and χ represent a bordism class $\{M,\chi\} \in \Omega_3(B\mathbb{Z}_m) = \mathbb{Z}_m$. Thus an odd multiple of (M,χ) is the boundary of a pair (W,χ'). We may define $w(M,\chi) \in L_0(\mathbb{Q}(\zeta_m)) \otimes \mathbb{Z}_{(2)}$ by imitating the definition of $\sigma(M,\chi) \in L_0(\mathbb{Q}(\zeta_m)) \otimes \mathbb{Q}$ in [2]. (Actually, there are two equivalent definitions given there. See [2], p. 186 where two intersection forms on the same vector space over $\mathbb{Q}(\zeta_m)$ differ by a factor of m. As each prime which divides m is a norm (see the proof of Lemma 3), these forms represent the same element in $L_0(\mathbb{Q}(\zeta_m)) \otimes \mathbb{Z}_{(2)}$.) Note that if $j: \mathbb{Z}_m \to \mathbb{Z}_{nm}$ is an injection then it induces an injection

$$j_{\#}: L_0(Q(\zeta_m)) \otimes Z_{(2)} \to L_0(Q(\zeta_{nm})) \otimes Z_{(2)}$$

by the argument of [2], p. 194. Hence we have $j_{\#}(w(M,\chi)) = w(M, j\circ\chi)$. We will let $\delta_m(M,\chi) \in N_m$ be $\prod_{i=1}^{\frac{1}{2}(m-1)} \operatorname{dis} w(M, i\chi)$.

Let $L(m,q)$ denote the Lens space given by $(-m/q)$-surgery on the unknot, and let $J(m/q)$ denote (m/q)-Dehn-surgery to S^3 along J. In either case, for any n dividing m, let χ_n be the homomorphism that sends a meridian to the residue class of 1 in $\mathbb{Z}_n$. In Section 5 we will derive formulae for $\operatorname{dis}(w(L(m,q), i\chi_m))$ and will show that, for m' a divisor of m, $\delta_{m'}(L(m,q), \chi_{m'})$ is (up to sign) a norm. In fact it is given by the Legendre–Jacobi symbol $(\frac{q}{m'})$.

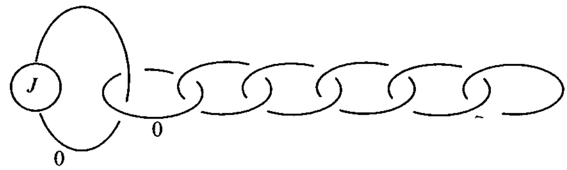

Fig. 4

THEOREM 8. $w(J(m/q), i\chi_m) = w_{i/m}(J) - w(L(m,q), i\chi_m)$.

Proof. $L(m,q)$ is the boundary of a plumbing with weights given by the coefficients of a continued fraction expansion of m/q (see [**14**], p. 70). Thus we can think of $-L(m,q)$ as integral surgery on a link which is a linear chain of unknots. The character $i\chi_m$ sends the meridian of the first component to the residue class of i. Moreover $(J(m/q), i\chi_m)$ is given by this same surgery description, except that the first component is now J which the second component links simply. The character $i\chi_m$ again sends the meridian of the first component to the residue class of i. Let (M,χ) denote the disjoint union of $(J(0), i\chi_m)$ and $(-L(m,q), i\chi_m)$. We can build a cobordism which the characters extend over from (M,χ) to $(J(m/q), i\chi)$ as follows. First we obtain W_1 by adding a 1-handle to $I \times (M,\chi)$ along $\{1\} \times (M,\chi)$ so that the new boundary component is $(J(0), i\chi_m) \# (-L(m,q), i\chi_m)$. This last manifold has a surgery description given by the above chain of unlinks and off to the side the knot J with framing zero. Now we wish to add a 2-handle to W_1 and obtain W_2. The 2-handle is attached with framing zero in the above surgery description along an unknot which simply links the first component in the chain of unlinks and also simply links the knotted component off to the side. See Figure 4. Note that the character vanishes on the attaching curve of the handle. We may identify the new boundary component M' with $(J(m/q), i\chi)$ using Kirby calculus [**17**]. First we slide the 2-handle attached along the first unknot in the chain over the 2-handle attached along J. We are left with the above mentioned linear chain surgery description of $(J(m/q), i\chi_m)$ together with an extra copy of J with framing zero accompanied by a meridian of J also framed zero. As is well-known, we can use this meridian to unknot this copy of J and unlink it from the rest of the diagram. Thus we could just as well have a Hopf link framed zero off to the side. Removing the Hopf link does not affect the 3-manifold. Thus we obtain a framed link identical to the original description of $-L(m,q)$, except that the first component is now the knot J instead of an unknot. The same argument which identifies $-L(m,q)$ with surgery on the chain of unlinks, now identifies M' with $J(m,q)$.

Now we consider the chain complex for (W, M) with $\mathbb{Q}(\zeta_d)$ coefficients twisted by our character. The first and second chain groups are copies of $\mathbb{Q}(\zeta_d)$, and ∂_1 is an isomorphism. Thus the twisted homology of (W,M) is zero. It now follows from the long exact sequence for the pair (W,M) that the twisted homology of M maps onto that of W, and hence the intersection pairing (with twisted coefficients) is trivial. Since the 2-handle was attached with framing zero, the ordinary intersection form of W is also identically zero. As the boundary of W is the disjoint union of

$$(J(m/q), i\chi_m),\ -(J(0), i\chi_m) \quad \text{and} \quad (L(m,q), i\chi_m), \quad \text{and} \quad w(J(0), i\chi_m) = w_{i/m}(J)$$

(see 'calculating τ' in section 3 of [**10**]), the result follows. This is much like the proof of additivity of $\tau(K,\chi)$ in [**7**]. ∎

If there is a 4-manifold W such that the boundary of W is the disjoint union of M_0 and $-M_1$ and such that the inclusion of each M_i into W induces an isomorphism on $\mathbb{Z}_p$-homology, we say that M_0 and M_1 are $\mathbb{Z}_p$-homology cobordant, and that W is a $\mathbb{Z}_p$-homology cobordism between M_0 and M_1. If X is a space and $\chi \in H^1(X, \mathbb{Z}_d)$, let X_χ denote the associated d-fold covering space of X. Let $\eta(M,\chi)$ be the dimension of $H_1(M, \mathbb{Q}(\zeta_d))$ with twisted coefficients given by χ. This is the same as the nullity invariant defined in [**6**]. Even the signature version of the following theorem is new if $\eta(M,\chi)$ is non-zero. For related results see Ruberman [**26**], theorem 1·2 and Matic [**23**], theorem 7·1.

THEOREM 9. *If W is a $\mathbb{Z}_p$-homology cobordism between M_0 and M_1, and $\chi \in H^1(W, \mathbb{Z}_d)$ where d is a power of p, then W_χ is a $\mathbb{Z}_p$-homology corbordism. If $\chi_i \in H^1(M_i, \mathbb{Z}_d)$ denotes the restrictions of χ, then $w(M_0,\chi_0) = w(M_1,\chi_1)$, and $\eta(M_0,\chi_0) = \eta(M_1,\chi_1)$.*

Proof. Let α_i denote the inclusion of M_{i_χ} in W_χ. We consider the exact sequence of Smith homology groups, (1·2) in [**6**], for both M_{i_χ} in W_χ. As this sequence is natural, we obtain commutative diagrams for each $0 < k \leqslant d$. Using $H^\delta(X_\chi) = H(X, \mathbb{Z}_p)$ and the 5-lemma, we may prove inductively that the α_i induce isomorphisms on $H^{\delta^k}(M_{i_\chi}) \to H^{\delta^k}(W_\chi)$ for all $0 < k \leqslant d$. As in general $H^{\delta^d}(X_\chi) = H(X_\chi, \mathbb{Z}_p)$, this means that W_χ is a $\mathbb{Z}_p$-homology cobordism. Therefore W_χ is also a $\mathbb{Q}$-homology cobordism. As all the 2-dimensional homology of W_χ comes from the boundary, the intersection form on W_χ is identically zero. Thus the eigenspace signatures of W_χ are zero. This gives the equality of Witt invariants. The equality of nullities follows as each $H_1(M_{i_\chi}, \mathbb{Q})$ is equivariantly isomorphic to $H_1(W_\chi, \mathbb{Q})$. ∎

By a $\mathbb{Z}_p$-homology lens space we mean a closed 3-manifold M with $H_1(M, \mathbb{Z}_p)$ isomorphic to $\mathbb{Z}_p$. By a proof similar to that of (3·5) of [**9**], but concentrating on the p-primary part of $H_1(M_0 \# -M_1)$, we have

THEOREM 10. *If M_0 and M_1 are $\mathbb{Z}_p$-homology lens spaces and there is a locally flat embedding of $M_0 \# -M_1$ in a homology 4-sphere, then there is a $\mathbb{Z}_p$-homology cobordism W from M_0 to M_1. Moreover the inclusions of M_i to W induce isomorphisms on the p-primary part of the first homology.*

Thus we have the following generalization of (3·3) of [**9**]:

COROLLARY 11. *Let d divide m, and N be any $\mathbb{Z}_d$-homology sphere and suppose that $J_1(m/q_1) \# -J_1(m/q_2) \# N$ embeds locally flat in a homology 4-sphere. Then there is a c such that $q_2 = c^2 q_1 \bmod d$, and such that $w(J_1(m/q_1), i\chi_d) = w(J_2(m/q_2), ic\chi_d)$ for $0 < i < d$.*

Now we can give the proof of Corollary 2 in the Introduction. By Theorem 8 and Corollary 16 in Section 4, $\delta_d(J_i(m/q_i), \chi_d) = \pm \Delta_d(J_i)$ in N_d. By Corollary 11,

$$w(J_1(m/q_1) \# N, i\chi_d) = -w(J_2(m/q_2), ic\chi_d).$$

It is not hard to see that

$$w(J_1(m/q_1) \# N, i\chi_d) = w(J_1(m/q_1), i\chi_d).$$

Thus $\operatorname{dis}(w(J_1(m/q_1), i\chi_d)) \operatorname{dis}(w(J_2(m/q_2), ic\chi_d))$ is a norm. We take the product of these equations as i varies from 1 to $(d-1)/2$, and make use of the fact that the set $\{c, 2c, 3c, \ldots, ((d-1)/2)c\}$ contains exactly one element from each set $\{s, -s\}$ for s between 1 and $d-1$. As $w(M, -\chi)$ equals $w(M,\chi)$, we obtain

$$\delta_d(J_1(m/q_1), \chi_d)\, \delta_d(J_2(m/q_2), \chi_d) = 1$$

in N_d. So $\Delta_d(J_1)\Delta_d(J_2)$ is a d-norm.

Example. By Theorem 14 below, $\Delta_3(T_{18}) = (5)(11)$. As $5 = -1 \bmod 3$, which has even order in $\mathbb{Z}_3^*$, it follows that $T_{18}(3/q)\,\#\,L(3/q')$ does not embed in S^4. As T_{18} has trivial Arf invariant, the μ-invariant of $T_{18}(3/q)\,\#\,L(3,q)$ is trivial (see [13], theorem 2) and so does not provide an obstruction to embedding.

4. *Computing* $\Delta_m(J)$

Let J be a simple $(2n-1)$-knot in S^{2n+1}, and F a $(n-1)$-connected Seifert manifold for J with Seifert matrix V. Recall that $S = (V^t + (-1)^n V)$ is the invertible matrix over $\mathbb{Z}$ which gives minus the intersection pairing on V. Let $G = S^{-1}V^t$. By generalizing the argument in section 1 of [8], we obtain:

PROPOSITION 12. *$G^m - (G-I)^m$ is a presentation matrix for $H_n(B_m(J))$.*

This presentation is related to one given by Seifert [27] for classical knots. However Seifert's presentation requires that V be given with respect to a symplectic basis for $H_1(F)$. The following proposition generalizes the formula of Fox mentioned in Section 1, while giving an alternative method of proving it.

PROPOSITION 13. *The order of $H_n(B_m(J))$ is $|\prod_{i=1}^{m-1}\Delta_J(\zeta_m^i)|$. So for m odd, the order of $H_n(B_m(J))$ is $(\Delta_m(J))^2$. Here, the order of an infinite abelian group is taken to be zero.*

Proof. Note that $G-1 = (-1)^{n+1}S^{-1}V$. Making use of the fact that G commutes with $G-I$, we can factor $G^m-(G-I)^m$ as

$$\prod_{i=1}^{m} G-\zeta_m^i(G-I) = S^{-m}\prod_{i=1}^{m}(V^t+(-1)^n\zeta_m^i V).$$

It is easy to check that the determinant of the last expression is $\prod_{i=1}^{m-1}\Delta_J(\zeta_m^i)$. ∎

Let T_k denote the k-twist knot illustrated in Figure 2 with Seifert matrix and Alexander polynomial given by

$$V = \begin{pmatrix} k & 1 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \Delta_{T_k}(t) = -kt+(2k+1)-kt^{-1}.$$

We also let T_k denote the simple $(2n-1)$-knot with n odd with the same Seifert pairing. All the signatures of T_k vanish if k is non-negative, and so T_k represents torsion in Levine's group. It is easy to see T_k is algebraically slice if and only if k has the form $r(r+1)$. One may also see that $w_{i/m}(T_k)$ is zero if m is odd and k has the form r^2 as follows. According to [1] or [5], a form with even rank, trivial discriminant and all signatures zero is trivial. As m is odd, we have $Q(\zeta_m)$ contains $\omega = -(\zeta_m^i)^{\frac{1}{2}(m-1)}$, a square root of ζ_m^i. If we let $z = -r\omega+1+r\varpi$, then the norm of z is $\Delta_{T_{r^2}}(\zeta_m^i)$.

Let U_k denote a simple $(2n-1)$-knot with n even and with $\Delta_{U_k}(t) = \Delta_{T_1}(t)\Delta_{T_v}(t)$ where $v = 5k^2+5k+1$. By [18], proposition 1 there is such a knot. According to [19], §23, U_k represents a torsion class in the knot cobordism group. Note that $\Delta_m(U_k) = \Delta_m(T_1)\Delta_m(T_v)$. So if m is odd and d divides m, then $\Delta_m(U_k)$ is a d-norm if and only if $\Delta_m(T_v)$ is a d-norm.

THEOREM 14. *Let* $\lambda = \frac{1}{2}(1+\sqrt{(4k+1)})$ *and* $\bar{\lambda} = \frac{1}{2}(1-\sqrt{(4k+1)})$.

(i) *For m odd,* $H_n(B_m(T_k)) = \mathbb{Z}_{\lambda^m+\bar{\lambda}^m} \oplus \mathbb{Z}_{\lambda^m+\bar{\lambda}^m}$ *and hence* $\Delta_m(T_k) = \lambda^m+\bar{\lambda}^m$.

(ii) *For all m,* $\lambda^m+\bar{\lambda}^m = a_m$, *where the a_i are defined recursively by* $a_0 = 2$, $a_1 = 1$, *and* $a_i = ka_{i-2}+a_{i-1}$.

Proof. The homology of the m-fold cover is presented by the matrix $G^m-(G-I)^m$ where

$$G = (V^t-V)^{-1}V^t = \begin{bmatrix} 1 & -1 \\ -k & 0 \end{bmatrix}.$$

Now G has eigenvalues λ and $\bar{\lambda}$, and hence on diagonalizing we see that $G^m-(G-I)^m$ is conjugate to the diagonal matrix with $\lambda^m-(\lambda-1)^m$ and $\bar{\lambda}^m-(\bar{\lambda}-1)^m$ on the diagonal. Noting that $(\lambda-1) = -\bar{\lambda}$ and $(\bar{\lambda}-1) = -\lambda$, it follows that for m odd, P is conjugate to, and hence equal to, the diagonal matrix with $\lambda^m+\lambda^m$ on the diagonal.

(For m even one can keep track of the conjugation and find that the homology is the direct sum of a cyclic group of order $(\lambda^m-\bar{\lambda}^m)/(\lambda-\bar{\lambda})$ and a cyclic group of order $(4k+1)(\lambda^m-\bar{\lambda}^m)/(\lambda-\bar{\lambda})$.)

For (ii) one considers the generating function

$$f(t) = \sum_{j=0}^{\infty}(\lambda^j+\bar{\lambda}^j)t^j.$$

This can be written as a rational function via the geometric series, and we find that $f(t) = 1/(1-\lambda t)+1/(1-\bar{\lambda}t)$. Writing this over a common denominator gives $f(t) = (2-t)/(1-t-kt^2)$. Finally one has

$$(2-t) = (1-t-kt^2)\left(\sum_{j=0}^{\infty}(\lambda^j+\bar{\lambda}^j)t^j\right),$$

and the recursion relation follows. The first two terms a_0 and a_1 are computed explicitly. This recursion formula does not give the order of the homology for m even. ∎

Calculations. We wish to apply Corollary 1 in the case that $m(K) = 1$, and also Corollary 6 when $\mu = 1$. Let C be the collection of prime powers which divide 2^q-1 for some odd prime power q. If d is in C, the order of 2 in $(\mathbb{Z}_d)^*$ must be an odd prime power. Of course, the converse also holds. The only elements of C less than one hundred are 7, 23, 31, 47, 73, and 89. The corresponding q's are 3, 11, 5, 23, 9, and 11. So for instance the knot in Figure 3 with J taken to be T_k will fail to be slice, if $\Delta_d(T_k)$ fails to be a d-norm for some d in C. In this case, $L_1(T_k)$ will not be concordant to a boundary link.

For k of the form $r(r+1)$ or r^2, $\Delta_d(T_k)$ must be a d-norm for all d. For all k with $0 < k < 45$, except those of the form $r(r+1)$ or r^2, we have been able to check that $\Delta_d(T_k)$ is not a d-norm for some d in C. The calculations are lengthy. For instance, $\Delta_7(T_{10})$ is a 7-norm, but $\Delta_{23}(T_{10})$ has a prime factor of 26633, which is congruent -1 modulo 23, and so $\Delta_{23}(T_{10})$ is not a 23-norm. $\Delta_d(T_{45})$ is a d-norm for $d = 7$, 23, 31, or 47. $\Delta_{73}(T_{45})$ is a 63 digit composite integer which we have been unable to factorize.

We have checked that $\Delta_7(U_1)$, $\Delta_7(U_2)$ and $\Delta_7(U_5)$ are not 7-norms. Also $\Delta_{31}(U_3)$ and $\Delta_{31}(U_4)$ are not 31-norms. Thus $L_1(U_k)$ is not concordant to a boundary link for $k = 1$, 2, 3, 4 and 5. Our calculations were done using MAPLE running on a DEC computer and using Mathematica running on a NeXT computer.

Remarks. If p is a prime, then $G^{p^r}-(G-I)^{p^r} = I \bmod p$, and so the order of $H_n(B_{p^r}(J))$ is congruent to 1 modulo p. Here we need to observe that for m odd, $\prod_{i=1}^{m-1}\Delta_J(\zeta_m^i)$ is positive and it is the determinant of $G^{p^r}-(G-I)^{p^r}$, by the equation in the proof of Proposition 13. Alternatively, letting d denote a power of an odd prime d, we can note that $\mathbb{Z}[\zeta_d]/(1-\zeta_d)\mathbb{Z}[\zeta_d]$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$, as p is $u(1-\zeta_d)^{\phi(p^r)}$ where u is a unit in $\mathbb{Z}[\zeta_d]$. Thus

$$\prod_{i=1}^{d-1}\Delta_J(\zeta_d^i) \equiv \prod_{i=1}^{d-1}\Delta_J(1) \equiv 1 \bmod p$$

as $\Delta_J(1)$ is ± 1.

If n is odd, then $\Delta_J(1) = 1$. Thus, by the same argument $\prod_{i=1}^{\frac{1}{2}(d-1)}\Delta_J(\zeta_d^i) \equiv 1 \bmod p$. As $\Delta_d(J)$ is the absolute value of this number, $\Delta_d(J)$ is $1 \bmod p$ if and only if $\operatorname{dis}(\sum_{i=1}^{\frac{1}{2}(d-1)} w_{i/d}(J))$ is positive. For a form of even rank, the dis is positive if and only if the signature is zero mod 4. Thus $\Delta_d(J) \equiv 1 \bmod p$ if and only if $\sum_{i=1}^{\frac{1}{2}(d-1)}\sigma_{i/d}(J) \equiv 0$ mod 4. Similarly $\Delta_d(J) \equiv -1 \bmod p$ if and only if $\sum_{i=1}^{\frac{1}{2}(d-1)}\sigma_{i/d}(J) = 2 \bmod 4$. It is not hard to see that $\prod_{i=1}^{d-1}\sigma_{i/d}(J)$ is the signature of the d-fold branched cyclic cover of the $2n$-disk along a pushed in copy of F. If n is odd, then $\sum_{i=1}^{d-1}\sigma_{i/d}(J) = 2\sum_{i=1}^{\frac{1}{2}(d-1)}\sigma_{i/d}(J)$. Thus if $n = 1$, then $\mu(B_d(J))$, the Rochlin invariant of $B_d(J)$, is congruent to $\sum_{i=1}^{d-1}\sigma_{i/d}(J) \bmod 16$. So for a classical knot J we see that $\Delta_d(J) \equiv 1 \bmod p$ if and only if $\mu(B_d(J)) \equiv 0 \bmod 8$. Similarly $\Delta_d(J) \equiv -1 \bmod p$ if and only if $\mu(B_d(J)) \equiv 4 \bmod 8$.

5. *Computing* dis $(w(L(m,q),\chi))$

Let m be odd, $\zeta = \zeta_m$, and define

$$a_{i,j} = (1-\zeta^i)(1-\zeta^j), \quad b_{i,j} = a_{i,j} - \overline{a_{i,j}}, \quad c_{i,j} = (\zeta-\bar{\zeta})\, b_{i,j} \quad \text{and} \quad e(\alpha) = \prod_{j \neq -\alpha} c_{\alpha,j}.$$

Here and below, i, j, α, and β are elements of $\mathbb{Z}_m\backslash\{0\}$. In general, for $q \in \mathbb{Z}_m^*$, we let $\hat{q}$ denote the inverse of q in $\mathbb{Z}_m^*$.

THEOREM 15. *$w(L(m,q), s\chi_m)$ has odd rank and has discriminant* $e(s)\,e(s\hat{q})$.

Proof. Consider the manifold with action (B,T) where B is the m-fold branched cyclic cover of S^2 along m points together with the covering transformation T whose action at each of the m fixed points is rotation by an angle of $2\pi/m$. Note that the 1-eigenspace for the action of T on $H_2(B,Q(\zeta))$ is trivial. The Witt class of the intersection form on the ζ^α-eigenspace of (B,T) is given by the intersection form on the ζ^α-eigenspace of B with m equivariant disjoint discs removed modulo its radical. Let $E(\alpha)$ denote this form. By the proof of (5·2) in [**6**], $E(\alpha)$ is represented by a $(m-2)\times(m-2)$ diagonal matrix with entries $-b_{\alpha,j}$ for $j \neq -\alpha$ on the diagonal. The product $(B,T)\times(B,T^q)$ has m^2 fixed points with local action at the fixed points given by multiplication by (ζ,ζ^q). If we delete neighbourhoods of these fixed points and reverse the orientation on the result, we obtain a equivariant null bordism of m^2 copies of $(L(m,q),\chi)$. So

$$d^2 w(L(m,q), s\chi) = [H(s)] - [H(0)],$$

where [ ] denotes Witt class and $H(a)$ denotes the intersection form on the ζ^a-

eigenspace of $-(B,T)\times(B,T^q)$. (See the discussion preceding proposition 1·1 in [6] for an explanation of why this holds for s not equal to one.) Moreover we have

$$H(s) = \bigoplus_{\alpha+q\beta=s} E(\alpha)\otimes E(\beta) = \bigoplus_{\beta\neq s\hat{q}} E(s-q\beta)\otimes E(\beta)$$

and
$$H(0) = \bigoplus_{\alpha+q\beta=0} E(\alpha)\otimes E(\beta) = \bigoplus_{\beta} E(-q\beta)\otimes E(\beta).$$

The reason why we do not need to insert a minus sign to account for the minus in front of $(B,T)\times(B,T^q)$ is the equation

$$(a\times b)\cup(c\times d) = (-1)^{\deg(b)\deg(c)}((a\cup c)\times(b\cup d)).$$

We focus first on calculating the determinant of the form on $E(\alpha)\otimes E(\beta)$ modulo norms in $\mathbb{Q}(\zeta+\bar{\zeta})$. Note that $c_{i,j}$ and $e(\alpha)$ are in $\mathbb{Q}(\zeta+\bar{\zeta})$. The determinant of the form on $E(\alpha)\otimes E(\beta)$ is given by

$$\prod_{\substack{i\neq-\alpha\\ j\neq-\beta}} b_{\alpha,i}b_{\beta,j} = (\prod_{i\neq-\alpha} b_{\alpha,i})^{m-2}(\prod_{i\neq-\beta} b_{\beta,i})^{m-2} = ((\zeta-\bar{\zeta})^2)^{-(m-2)^2}e(\alpha)^{m-2}e(\beta)^{m-2},$$

which is congruent to $-(e(\alpha))(e(\beta))$ modulo norms. Here we make use of the fact $(\zeta-\bar{\zeta})^2\equiv-(\zeta-\bar{\zeta})(\bar{\zeta}-\zeta)\equiv-1$ modulo norms. Note that the rank of this form is $(m-2)^2$.

Now we work out the determinant of the form on $H(s)$ modulo norms. It is given by $-\prod_{\beta\neq s\hat{q}}e(s-q\beta)\,e(\beta) = -e(s)\,e(s\hat{q})$. Here we have observed that the first product is almost $(\prod_\beta e(\beta))^2$. The rank of the form is $(m-2)^3$.

We also need the determinant and rank of the form on $H(0)$. It has rank

$$(m-2)^2(m-1)$$

and its determinant is a norm by a calculation similar to that above for $H(1)$. Thus the form on $H(1)$ *direct sum minus the form on* $H(0)$ has rank $(m-2)^2(2m-3)$ which is always congruent 3 modulo 4 and has determinant $-e(s)\,e(s\hat{q})$ modulo norms. Thus the discriminant is $e(s)\,e(s\hat{q})$ and $m^2w(L(d,q),\chi^s)$ has odd rank and discriminant $e(s)\,e(s\hat{q})$. As $m^2\equiv1 \bmod 4$, we see that $w(L(d,q),\chi^s)$ has the same discriminant and rank. ∎

Remark. If we let m' be a divisor of m, then $w(L(m,q),s\chi_{m'})$ still has discriminant $e'(s)\,e'(s\hat{q})$. Here $e'(\alpha)$ is defined by letting $\zeta=\zeta_{m'}$, and letting i, j, α be elements of $\mathbb{Z}_{m'}\backslash\{0\}$ in the formulae for $e(\alpha)$. This follows from the naturality of $w(M,\chi)$ under composition of χ with a map $j:\mathbb{Z}_{m'}\to\mathbb{Z}_m$, as discussed at the beginning of Section 3.

COROLLARY 16. *If m' is a divisor of m, then $\delta_{m'}(L(m,q),\chi_{m'})$ is given by the Legendre–Jacobi symbol $(\frac{q}{m'})$.*

Proof. Note that $a_{-i,-j}=\overline{a_{i,j}}$, $b_{i,j}=-b_{-j,-i}$, and so $c_{i,j}=-c_{-j,-i}$. Thus

$$e(-\alpha) = \prod_{j\neq\alpha} c_{-\alpha,j} = \prod_{-j\neq\alpha} c_{-\alpha,-j} = -\prod_{-j\neq\alpha} c_{\alpha,j} = -e(\alpha).$$

Now
$$\delta_{m'}(L(m',q),\chi_{m'}) = \prod_{s=1}^{\frac{1}{2}(m'-1)} e(s)\,e(s\hat{q}).$$

Notice that the sets $\{1,2,\ldots,\frac{1}{2}(m'-1)\}$ and $\{\hat{q},2\hat{q},\ldots,\frac{1}{2}(m'-1)\,\hat{q}\}$ both contain exactly one element from each set of the form $\{s, -s\}$. As $e(-\alpha) = -e(\alpha)$, and $e(\alpha) \in \mathbb{Q}(\zeta_{m'} + \bar{\zeta}_{m'})$, the above product is a power of -1. The precise power is simply the number of elements in $\{\frac{1}{2}(m'-1), 2, \ldots, m'-1\} \cap \{\hat{q}, 2\hat{q}, \ldots, \frac{1}{2}(m'-1)\hat{q}\}$. By the generalized Gauss Lemma ([15], p. 130), this power of -1 is just the Legendre–Jacobi symbol $(\frac{\hat{q}}{m'})$. It is easy to see that $(\frac{\hat{q}}{m'}) = (\frac{q}{m'})$. ∎

This work was supported in part by a grant from the NSF. The authors would like to thank Jim Davis, John Ewing, Rick Litherland, Robert Perlis and Neal Stoltzfus for assistance, particularly with the algebraic number theory.

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